Critique of Previous Comprehensive Studies of Self-referential Paradoxes

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Abstract: *The fact that it produces a paradox implies the possibility that a theory or a thought is not entirely rational. At the same time, the possible resolution of paradoxes suggests candidates for the conditions that may make theories or thoughts consistent. Many scholars have investigated paradoxes to find their solutions. The primary approach to a paradox is to consider each solution in turn (the one-by-one approach); studies using this approach have resulted in many diverse solutions, but it is limited by serious problems. Hence, I will adopt another approach, termed here as the comprehensive approach, to consider a general solution or solutions to all paradoxes. Here the focus will be on self-referential paradoxes. A self-referential paradox indicates a paradox that occurs when the subject of a proposition is in part the proposition itself. Further, some paradoxes are not self-referential but seem to be. To preserve the appearances of our intuition of a self-reference, I introduce a categorization of self-references into two types (with two corresponding types of accompanying paradoxes): the first is a narrow self-reference, that is, precisely a reference to itself (this matches the existing characterization). The other type is a broad self-reference or a reference to a group or groups containing itself.*

In this paper, first, I will examine the works of three figures, Bertrand Russell; Graham Priest; and Martin Pleitz, who take a comprehensive approach: considering the general conditions for self-referential paradoxes, analyzing them, and suggesting general solutions for them. Their conclusions are valuable to a certain extent, but their works are limited to addressing only narrow self-referential paradoxes. In this paper, I will exhibit two broad self-referential paradoxes that cannot be accounted for by these works.

0. Introduction

Paradox is, as Sainsbury mentions, "an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises".[1] The existence of paradox implies that our theories or thoughts may not be rational. At the same time, solution(s) to paradoxes can suggest candidates for conditions for our theories or thoughts to be rational. This is why study of paradox is significant.

Among various paradoxes, I concentrate in this paper on self-referential paradoxes. First, I review previous studies from the "comprehensive approach" (explained below) to self-referential paradoxes. Then, we see that such studies cannot cover some self-referential paradoxes.

1. Preliminaries

Before approaching the main subject, let me explain what I address here and how I do so.

1.1 Characterization of Self-Referential Paradoxes

In this paper about self-referential paradoxes, I characterize them as paradoxes caused by referring to themselves or groups that contain themselves. As you may know, a self-referential paradox is characterized usually as "a paradox caused by referring to itself", or self-reference is characterized only as "reference to itself". There are, however, paradoxes that actually do not refer to themselves, but seem to be self-referential (see section 3). In general, two attitudes can be taken toward what is not X in terms of existing criteria, but seems X-like. The first option is to judge it as not X; this attitude is important for conducting strict studies about X. The second is to suggest new criteria that fit our intuition better; this way, we may discover new information about (or at least related to) X. Their fruits differ, but both attitudes are important. In this paper I adopt the latter attitude, and hence, I adopt the characterization above of self-referential paradox. Further, let us call a reference to itself and groups that contain itself, respectively, the "narrow self-reference" and the

[1] Sainsbury, R.M. (2009), *Paradoxes*, Cambridge University Press, p. 3.

"broad self-reference", and let us define "narrow self-referential paradox" and "broad self-referential paradox" accordingly.[2]

1.2 Two Approaches to Paradoxes

The mainstream approach to paradoxes (including self-referential paradoxes) is to consider each solution(s) for each paradox. Let us call this approach the "one-by-one approach." True, this approach has produced various results, and it has two serious problems. First, the approach inclines to be ad hoc. Conceivably, then, when a new paradox is found, it cannot be solved by previously existing solutions. The second and crucial reason is that one solution to a paradox may be incompatible with other solution(s) to different paradox(es). This means that solutions may work at most for our particular theories or thoughts, but not for whole ones; thus, such solutions cannot be helpful for making our whole theories or thoughts rational. Finding a compatible set of solutions to each paradox seems to dissolve this problem. However, a new solution to a new paradox is not guaranteed to be compatible with an existing set of solutions to existing paradoxes.

There is yet another approach to paradoxes, called the "comprehensive approach", which attempts to find general solution(s) to all paradoxes, not individual solutions to individual paradoxes. In adopting this approach, you do not have to fear occurrences of new paradoxes because the solution is general to all paradoxes, so it should solve an emerging paradox if the solution is truly general. Moreover, you do not have to fear the incompatibility of each solution because the solution is, again, a general solution to all paradoxes, so it cannot be incompatible with a solution to another paradox. I adopt this approach to self-referential paradoxes.

You may doubt the comprehensive approach even to only self-referential paradoxes because this approach requires two premises: "all self-referential paradoxes have the same conditions", and "the conditions are essential for such paradoxes appearing". You may cast doubt on them, and, at least, F. Ramsey should oppose this idea. He divides self-referential paradoxes into two groups, A and B.

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[2] There is a paradox that is self-referential paradox-like, but seems to be neither a narrow nor a broad self-referential paradox; it is Yablo's paradox in Yablo, S. (1993), "Paradox without self-reference", *Analysis*. Because of this paradox, you may think my characterization cannot save our intuition about self-reference. But this paradox is in fact a narrow self-referential paradox. See Priest, G. (1997), "Yablo's Paradox", *Analysis*; and Beall, J.C. (2001), "Is Yablo's paradox non-circular?", *Analysis*.

> Group A consists of contradictions which, were no provision made against them, would occur in a logical or mathematical terms, ... But the contradictions of Group B are not purely logical and cannot be stated in logical terms alone; for they all contain some reference to thought, language, or symbolism, which are not formal but empirical terms. So they may be due not to faulty logic or mathematics, but to faulty ideas concerning thought and language.[3]

Thus Ramsey should think it not the case that "condition(s) of all self-referential paradoxes is the same". Hence, he and his followers should not admit the comprehensive approach to self-referential paradoxes. The reply to this idea is, however, not difficult. As Priest[4] states, Ramsey's distinction is superficial. His interest is what notions or vocabulary appear in the paradox, but it does not matter what conditions the paradox has. Therefore, such objection is not crucial.

2. Previous Studies

In this section, we examine previous studies about (narrow) self-referential paradoxes from the comprehensive approach. I concentrate especially on three figures: B. Russell (1905), G. Priest (2002), and M. Pleitz (2014). Other comprehensive studies address self-referential paradoxes, but only these three explicitly show conditions of self-referential paradoxes and suggest treatments by analyzing these conditions.

First of all, we examine work by Russell, a pioneer of this approach (section 2.1). The second figure is Priest, who shows more generalized conditions than Russell's (section 2.2). A certain self-referential paradox is a candidate for a counterexample of his work, so next, we check whether it is genuinely a counterexample (section 2.3). Then we see that the more general structure introduced by Pleitz can cover Curry's paradox (section 2.4).

2.1. Russell's Generalization

[3] Ramsey, F.P. (1978), *Foundations: Essays in Philosophy, Logic, Mathematics and Economics*, Routledge & Kegan Paul, p. 171.
[4] Priest, G. (2002), *Beyond the Limits of Thoughts*, Oxford University Press, p. 153.

Russell (1905) investigates some paradoxes related to transfinite numbers and detects that their essence is related not to mathematics, but to logic. I omit minute explanations of these paradoxes due to limited space. Here what matters is that they are self-referential paradoxes. Russell shows the general conditions of these paradoxes,[5] and Priest formalizes them to show them in a strict way.[6] However, Priest's formalization can be expressed in an even stricter way and a more refined manner, as I show in this paper.

> Given property φ and function δ:
> (1) $\exists\Omega(\Omega=\{y:\varphi(y)\}$
> (2) $\forall x(x\subseteq\Omega\rightarrow\neg(\delta(x)\in x))$
> (3) $\forall x(x\subseteq\Omega\rightarrow\delta(x)\in\Omega)$

Let us call these general conditions "Russell's Schema", following Priest. When these conditions are satisfied, contradiction is derived as follows: By (2) and (3),

$$\Omega\subseteq\Omega\rightarrow\neg(\delta(\Omega)\in\Omega)$$

and

$$\Omega\subseteq\Omega\rightarrow\delta(\Omega)\in\Omega$$

hold. Obviously, Ω itself is a subset of Ω; that is, $\Omega\subseteq\Omega$ holds. Hence, by *modus ponens*, both

$$\neg(\delta(\Omega)\in\Omega)$$

and

$$\delta(\Omega)\in\Omega$$

hold. Therefore, we can obtain a contradictory conclusion

$$\neg(\delta(\Omega)\in\Omega)\wedge\delta(\Omega)\in\Omega.$$

Needless to say, all paradoxes that Russell (1905) mentioned satisfy all three conditions of Russell's Schema.[7]

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[5] Russell, B. (2014), "On some difficulties in the theory of transfinite numbers and order types" in *The Collected Papers of Bertrand Russell,* volume 5, Routledge, p. 35.

[6] Priest, G. (2002), p. 129.

[7] By the way, Russell shows his solution, the theory of types, in other places: Russell, B. (1908) "Mathematical Logic as based on the Theory of Types", *American Journal of Mathematics*, Vol. 30, pp. 222–262. I omit his solution in this paper because it is not so compatible with our usages of ordinary languages. His solution may work well for mathematical paradoxes, but some self-referential paradoxes occur with our usual usage of language (see section 3).

2.2. Priest's Conditions and Solution

Russell's Schema works well for paradoxes he mentioned; however, it does not work well for some self-referential paradoxes. One such paradox is the most famous, the Liar paradox,[8] which occurs in the liar sentence φ, "φ is not true". To cover such paradoxes, Priest introduces more general conditions than Russell by generalizing them. Priest shows his conditions (the "Inclosure Schema")[9]; however, the conditions can be expressed in a stricter way. Hence, I refine his notations and show the refined Inlocure Schema. Needless to say, my refinement does not change the essence of Priest's formalization, only addressing the Inclosure Schema in a more logical manner.

Given properties φ and ψ, and function δ,
(1) $\exists\Omega(\Omega=\{y:\varphi(y)\}\wedge\psi(\Omega))$
(2) $\forall x(x\subseteq\Omega\wedge\psi(x)\rightarrow\neg(\delta(x)\in x))$
(3) $\forall x(x\subseteq\Omega\wedge\psi(x)\rightarrow\delta(x)\in\Omega)$

When these conditions are satisfied, we can also obtain a contradictory conclusion. By (2) and (3),

$$\Omega\subseteq\Omega\wedge\psi(\Omega)\rightarrow\neg(\delta(\Omega)\in\Omega)$$

and

$$\Omega\subseteq\Omega\wedge\psi(\Omega)\rightarrow\delta(\Omega)\in\Omega$$

are derived. By (1), $\psi(\Omega)$ holds. Because $\Omega\subseteq\Omega$ holds as mentioned, $\Omega\subseteq\Omega\wedge\psi(\Omega)$ holds. Therefore, both

$$\neg(\delta(\Omega)\in\Omega)$$

and

$$\delta(\Omega)\in\Omega$$

hold by *modus ponens*; that is, again, contradiction

$$\neg(\delta(\Omega)\in\Omega)\wedge\delta(\Omega)\in\Omega$$

is derived. Priest calls paradoxes satisfying these conditions (and causing this contradictory conclusion) "inclosure paradoxes".

We can easily confirm that the Inclosure Schema is more general than Russell's Schema, or, in other words, the latter is just a special case of the former.

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[8] Due to space limitations, I omit how some paradoxes are not covered by Russell's schema. See Priest (2002, pp. 143–144).
[9] Priest (2002), p. 134.

When a trivial property is chosen for ψ, ψ(Ω), and ψ(x) (for any *x*) trivially hold; then descriptions about ψ in the Inclosure Schema can be dismissed. When we omit such descriptions, we can obtain Russell's Schema.

Priest thinks one solution works sufficiently for all inclosure paradoxes—the "dialetheic" solution. Dialetheism is a philosophical position asserting that some contradictions are true.[10] That is, a solution Priest suggests can be expressed as "to accept a contradictory conclusion". To justify accepting contradictions, almost all dialetheists adopt a special logical system called "paraconsistent logic". Paraconsistent logic refers to a logical system in which the Law of Explosion

$$\alpha \wedge \neg\alpha \vdash \beta$$

does not hold. This law can be paraphrased as "if there is a contradiction, every sentence can be derived". This problem, "everything can be derived", is called triviality.[11] In a logical system in which the Law of Explosion holds—actually, it holds in most usual logics, like classical logic and intuitionistic logic—if there is only one contradiction, we cannot avoid falling into triviality, so contradiction is extremely harmful in such logics. On the other hand, if we adopt paraconsistent logic, triviality does not follow just from contradictions; paraconsistent logic makes contradictions harmless.

Let us see if formally, the Liar paradox can be solved by paraconsistent logic. The Liar paradox is caused by the liar sentence φ "this sentence is false", expressed as

$$\varphi := \neg\mathrm{Tr}[\varphi]$$

("Tr" is a truth predicate). Now, a contradictory conclusion is derived as follows:

$$\dfrac{\varphi \vee \neg\varphi \quad \dfrac{\dfrac{[\varphi]_1}{\mathbf{Tr}\ulcorner\varphi\urcorner}\,\text{TS} \quad \dfrac{[\varphi]_1}{\neg\mathbf{Tr}\ulcorner\varphi\urcorner}\,\text{def.}}{\mathbf{Tr}\ulcorner\varphi\urcorner \wedge \neg\mathbf{Tr}\ulcorner\varphi\urcorner}\,\wedge I \quad \dfrac{\dfrac{\dfrac{[\neg\varphi]_2}{\neg\neg\mathbf{Tr}\ulcorner\varphi\urcorner}\,\text{def.}}{\mathbf{Tr}\ulcorner\varphi\urcorner}\,DNE \quad \dfrac{[\neg\varphi]_2}{\neg\mathbf{Tr}\ulcorner\varphi\urcorner}\,\text{TS}}{\mathbf{Tr}\ulcorner\varphi\urcorner \wedge \neg\mathbf{Tr}\ulcorner\varphi\urcorner}\,\wedge I}{\mathbf{Tr}\ulcorner\varphi\urcorner \wedge \neg\mathbf{Tr}\ulcorner\varphi\urcorner}\,\vee I$$

[10] Apparently this position is very weird and unacceptable, but Priest himself tries to defend it. See Priest, G. (1998), "What is so Bad about Contradictions?", *The Journal of Philosophy*, Vol. 95, pp. 410¬426.

[11] There is a claim, called trivialism, that everything (every sentence) is true. For trivialists, triviality is not problematic. I will not consider this peculiar position in this paper. If you are interested in trivialism, see Kabay, P. (2008), *A Defense of Trivialism*, Ph.D. thesis, University of Melbourne.

(TS means T-Schema, and DNE means the rule of double negation elimination $\neg\neg\alpha \vdash \alpha$). Paraconsistent logic makes contradictions harmless, as I said. Hence, the Liar paradox, whose conclusion is a contradiction, can be solved by it.

2.3. Counterexample of Priest's Diagnosis

Paraconsistent logic seems to work well for all self-referential paradoxes because contradiction turns out to be innocuous. There is, however, a self-referential paradox whose conclusion is not a contradiction. This is Curry's paradox. Consider the Curry sentence φ, "if φ is true, then the moon is made of cheese". Suppose φ is true; that is, it is true that if φ is true, then the moon is made of cheese. Because both "φ is true" and "if φ is true, then the moon is made of cheese" hold, and we can obtain "the moon is made of cheese" by *modus ponens*. We inferred "the moon is made of cheese" by supposing "φ is true". This means that we proved "if φ is true, then the moon is made of cheese"; that is, we proved the sentence φ. Again, because φ is true (please note that it is now not a supposition, but proven fact), and "if φ is true, then the moon is made of cheese" is true (because of the truth of φ), we can derive "the moon is made of cheese" by *modus ponens*! In the same way, we can derive an arbitrary sentence, and we fall into triviality.

Let us see how this paradox causes triviality. The Curry sentence can be formalized as

$$\varphi := \mathrm{Tr}[\varphi]\to\psi$$

(ψ is an arbitrary sentence, like "the moon is made of cheese".) Now, ψ is derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi\to\varphi}{\varphi\to(\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi)}\,\text{def.}}{\mathbf{Tr}\ulcorner\varphi\urcorner\to(\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi)}\,\text{TS}}{\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi}\,\text{Contraction}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\varphi\to\varphi}{\varphi\to(\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi)}\,\text{def.}}{\mathbf{Tr}\ulcorner\varphi\urcorner\to(\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi)}\,\text{TS}}{\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi}\,\text{Contraction}}{\mathbf{Tr}\ulcorner\varphi\urcorner}\,\text{def.}}{\psi}\to E$$

In the case of inclosure paradoxes, their conclusions are contradictions. Paraconsistent logic separates contradiction from triviality, and hence, contradiction turns out to be harmless. In the case of Curry's paradox, however, its conclusion, triviality, is derived not through contradiction, but directly. So paraconsistent logic does not seem to work for this paradox.

How does Priest himself consider this paradox? In fact, he thinks it possible to consider it an inclosure paradox.

> For each paradox of this kind [= Curry's paradox], we can form a new paradox by replacing $\neg\alpha$ uniformly with $\alpha\rightarrow\beta$, where β is an arbitrary formula; or, more simply, with $\alpha\rightarrow\bot$, where $\bot$ is some logical constant entailing everything.[12]

> If [$\rightarrow$] is a material conditional then, in most logics, $\alpha\rightarrow\bot$ is logically equivalent to $\neg\alpha$, and so [Curry's paradox] is essentially the same as [inclosure paradoxes]. If, on the other hand, $\rightarrow$ is a non-material conditional $\cdots$, then $\alpha\rightarrow\bot$ and $\neg\alpha$ are quite different notions. . . . In this case, [Curry's paradox] belong[s] to a quite different family [than inclosure paradoxes].[13]

Priest's idea can be paraphrased as follows: Curry's paradox is counted as an inclosure paradox (and therefore, it can be solved by paraconsistent logic) if (i) you consider $\bot$ a logical constant entailing everything, and (ii) you define $\neg\alpha$ as $\alpha\rightarrow\bot$. Curry's paradox is not otherwise counted as an inclosure paradox. To check this, let us compare an inclosure paradox, the Liar paradox, to Curry's paradox. As above, the liar sentence and the Curry sentence are formalized as

$$\varphi := \neg Tr[\varphi]$$

and

$$\varphi := Tr[\varphi]\rightarrow\psi$$

respectively. When we accept (i), the Curry sentence is formalized as

$$\varphi := Tr[\varphi]\rightarrow\bot.$$

When we accept (ii), the liar sentence is

$$\varphi := Tr[\varphi]\rightarrow\bot$$

too. Now, formalization of Curry's paradox is the same as that of the Liar paradox. Therefore, we conclude that Curry's paradox can be counted as an inclosure paradox when you accept both (i) and (ii).

Both (i) and (ii) seem plausible. In my opinion, however, accepting both of them is impossible if you want to solve Curry's paradox by paraconsistent logic. As mentioned, the Law of Explosion $\alpha\wedge\neg\alpha\vdash\beta$ does not hold in paraconsistent logic. This law can be resolved into two parts:

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12 Priest (2002), p. 168.
13 Priest (2002), p. 169.

(a) $\alpha\wedge\neg\alpha \vdash \bot$

(b) $\bot \vdash \beta$.

Thus, to adopt paraconsistent logic—to reject the Law of Explosion—you should discard either (a) or (b). My diagnosis is that accepting both (i) and (ii) and discarding either (a) or (b) are incompatible. First, it is obvious that (i) and (b) are the same; both say that everything can be derived from ⊥. Second, when you admit (ii), (a) is derived as follows. Suppose that (ii) holds. Then $\alpha\wedge\neg\alpha$ is equal to $\alpha\wedge(\alpha\rightarrow\bot)$. Because ⊥ is derived from α and $\alpha\rightarrow\bot$,[14] it holds that $\alpha\wedge(\alpha\rightarrow\bot) \vdash \bot$; that is, (a) holds.

When you consider Curry's paradox an inclosure paradox—when you adopt (i) and (ii)—you should give up adopting paraconsistent logic—you cannot discard both (a) and (b); then you cannot use paraconsistent logic to solve any self-referential paradoxes (including Curry's paradox). When you do not consider Curry's paradox an inclosure paradox, it cannot be solved with a solution for inclosure paradoxes; that is, paraconsistent logic does not work for this paradox. In either case, another solution(s) than paraconsistent logic is required for Curry's paradox.

2.4. Pleitz's Structure and Solution

The last figure, Martin Pleitz, takes notice of this problematic paradox. He devises conditions that can cover both inclosure paradoxes and Curry's paradox[15] by modifying the Inclosure Schema. His structure, the Curry Schema, is as follows.

> Let φ and ψ be predicates and δ a function. Then the following threefold condition holds:
>
> (1) $\exists\Omega(\Omega=\{x|\varphi(y)\}\wedge\psi(\Omega))$
>
> (2) $\forall x(x\subseteq\Omega\wedge\psi(x)\rightarrow(\delta(x)\in x\rightarrow p))$
>
> (3) $\forall x(x\subseteq\Omega\wedge\psi(x)\rightarrow\delta(x)\in\Omega)$[16]

[14] In logical systems that do not admit *modus ponens*, this inference is invalid; I do not, however, take such systems into account because *modus* ponens is one of the most important rules of inference.

[15] It is not clear that he originally intended to consider conditions for both inclosure paradoxes and Curry's paradox; for him, his work is just for Curry's paradox, but it can cover both paradoxes.

[16] Pleitz, M. (2014), "Curry's Paradox and the Inclosure Schema", https://www.academia.edu/13030660, p. 9.

In (2), "*p*" refers to an arbitrary sentence. Now, let us see again the Inclosure Schema for comparison:

Given properties φ and ψ and a function δ,
(1) $\exists\Omega(\Omega=\{y:\varphi(y)\}\wedge\psi(\Omega))$
(2) $\forall x(x\subseteq\Omega\wedge\psi(x)\rightarrow\neg(\delta(x)\in x))$
(3) $\forall x(x\subseteq\Omega\wedge\psi(x)\rightarrow\delta(x)\in\Omega)$

We can easily observe that the Curry Schema is more general than the Inclosure Schema. The difference between them lies just in the second condition. As I mentioned, p in the consequent of the Curry Schema's second condition refers to an arbitrary one; that is, *p* can refer to ⊥. This means

$$\forall x(x\subseteq\Omega\wedge\psi(x)\rightarrow(\delta(x)\in x\rightarrow\bot))$$

is a special case of the Curry Schema's second condition. Moreover, the consequent of the Inclosure Schema's second condition $\neg(\delta(x)\in x)$ can be expressed as $\delta(x)\in x\rightarrow\bot$[17]; that is, the Inclosure Schema's second condition can be expressed as

$$\forall x(x\subseteq\Omega\wedge\psi(x)\rightarrow(\delta(x)\in x\rightarrow\bot));$$

this is the same as the formula above, so we can conclude that this is the special case of the Curry Schema's second condition.

In the case of Russell's Schema and the Inclosure Schema, their conclusion

$$\neg(\delta(\Omega)\in\Omega)\wedge\delta(\Omega)\in\Omega$$

is a contradiction; on the other hand, when the Curry's Schema's conditions are satisfied, triviality is derived. By (2) and (3),

$$\Omega\subseteq\Omega\wedge\psi(\Omega)\rightarrow(\delta(\Omega)\in\Omega\rightarrow p))$$

and

$$\Omega\subseteq\Omega\wedge\psi(\Omega)\rightarrow\delta(\Omega)\in\Omega$$

hold. $\Omega\subseteq\Omega$ holds, and by (1), $\psi(\Omega)$ holds too; so $\Omega\subseteq\Omega\wedge\psi(\Omega)$ holds. Therefore both

$$\delta(\Omega)\in\Omega\rightarrow p$$

and

$$\delta(\Omega)\in\Omega$$

hold. Therefore, by modus ponens, we can derive *p*; because p refers to anything, we fall into triviality.

[17] It holds that $\neg(\delta(x)\in x)$ is equal to $(\delta(x)\in x\rightarrow\bot)$ only when you define $\neg A$ as $A\rightarrow\bot$. Because this definition of negation is the most familiar, I do not here consider cases in which this equation does not hold.

Then, how can paradoxes covered by the Curry Schema be solved? Pleitz's suggestion is to adopt contraction-free logic,[18][19] a logical system without a rule, called contraction:

$\alpha\to(\alpha\to\beta)\vdash\alpha\to\beta$.

This rule holds in many usual logics, including classical logic, intuitionistic logic, and even some paraconsistent logics. Contraction does not appear explicitly in the argument above, but Pleitz finds the rule hidden in the argument[20] and shows that the conclusion should not be derived without contraction. Hence, contraction-free logic works well as a solution to paradoxes covered by the Curry Schema. In fact, in the proof of Curry's paradox, the contraction rule is used twice.

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi\to\varphi}{\varphi\to(\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi)}\text{def.}}{\mathbf{Tr}\ulcorner\varphi\urcorner\to(\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi)}\text{TS}}{\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi}\text{Contraction}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\varphi\to\varphi}{\varphi\to(\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi)}\text{def.}}{\mathbf{Tr}\ulcorner\varphi\urcorner\to(\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi)}\text{TS}}{\mathbf{Tr}\ulcorner\varphi\urcorner\to\psi}\text{Contraction}}{\mathbf{Tr}\ulcorner\varphi\urcorner}\text{def.}}{\psi}\to E$$

3. Paradoxes Excluded from Previous Works

In this brief summary of existing comprehensive studies of self-referential paradoxes, in considering their general conditions, we observed two solutions, paraconsistent logic and contraction-free logic. If they can solve every self-referential paradox, it follows that existing general conditions can cover every self-referential paradox. As mentioned, however, their work cannot solve some self-referential paradoxes. In this paper, I introduce two examples, the Ineffability paradox and the modified Berry's paradox.

3.1 Ineffability Paradox

[18] Pleitz's (2014) original suggestion is not contraction-free logic but contraction-free paraconsistent logic, although paraconsistency is redundant. In the autumn of 2017, I talked with Pleitz, and he admitted it.

[19] As mentioned, Priest suggests dialetheism for philosophical solution of inclosure paradoxes (adopting paraconsistent logic is the logical solution or logical interpretation of dialetheism); however Pleitz shows only the logical solution.

[20] To make use of the contraction visible, he investigates a modified version of the Curry Schema (pp. 11–12, 2014), which is essentially the same as the original Curry Schema.

3.1.1. What is the Ineffability Paradox?

The Ineffability paradox (the paradox of ineffability) is related to the profound and obscure notion "ineffability", usually meaning the impossibility of describing or expressing something. Many philosophers and religious figures in the East and West have mentioned this notion. In negative theology, it is said that God is ineffable. Lao Zi, one of the most influential thinkers in ancient China, says that the genuine Dao (Tao; 道) is ineffable. Conversion, enlightenment, qualia, and sense-data can be counted as ineffable too. On the other hand, the well-known Ineffability paradox occurs from mentioning ineffable thing(s). Once you state that x is ineffable (let us call such a statement an "ineffability statement"), x is expressed by "is ineffable". This means that x is not ineffable now; therefore, the contradictory conclusion "x is ineffable and x is not ineffable" is derived.

3.1.2. Is It a Self-referential Paradox?

You may hesitate to consider this paradox self-referential according to two doubts: "whether it is a paradox" and "whether it is a self-referential paradox".

3.1.2.1. Whether It Is a Paradox

One of the most apparently famous paradoxes is the Epimenides paradox. The Cretan philosopher Epimenides says, "All Cretans are liars". From his statement, it seems that the contradictory conclusion "All Cretans are liars, but one Cretan (that is, Epimenides) speaks truth" is derived. However, the Epimenides paradox is not in fact a paradox. If you suppose his statement is false, such a contradiction does not appear. You may think that the Ineffability paradox is not a paradox in the same sense. If you suppose the ineffability statement is false, the contradiction above does not appear.

This idea is true, but it is highly difficult philosophically to reject all ineffability statements. To deny "*x* is ineffable", for every *x*, you should insist that everything is effable. As Andre Kukla[21] argues, we are restricted epistemologically, so there should be some things we cannot recognize. We cannot mention such things, so they are ineffable. Therefore, some ineffability statements should be true.

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[21] Kukla. A, (2005), *Ineffability and Philosophy,* Routledge, pp. 53–58.

3.1.2.2. Whether It Is a Self-referential Paradox

Even if you adopt the Ineffability paradox as a paradox, there is another question: Is it a self-referential paradox? Let us compare this paradox with the Liar paradox. The liar sentence φ says "φ is not true"; this φ refers to the sentence φ itself. On the other hand, the ineffability statement says, "*x* is ineffable"; here is no reference to the statement itself.

True, in the Ineffability paradox, there is no such self-reference. Also true, however, is that this paradox apparently seems to be a self-referential paradox (at least for some people). "*x* is ineffable" (let us name the sentence p) implies "*x* is not ineffable" ($\neg p$); or, you may say p means $\neg p$. It is naïve, but not weird to think the Ineffability paradox is self-reference paradox-like.

I count as self-reference the reference to groups that contain itself—I suggested the new characterization of self-reference, or "broad self-reference", to preserve this intuition. In fact, the Ineffability paradox is a broad self-referential paradox. Let us check; "*x* is ineffable" implies that it does not hold "*x* is ineffable", "*x* is white", "*x* is fluffy", and so on. "is ineffable" implies rejection of any expression about *x*; that is, "is ineffable" refers to negations of any expression about *x*, including "*x* is ineffable".[22]

3.1.3. How It Slips from Existing Works

It seems that this paradox can be solved by existing solutions, especially by paraconsistent logic, because its conclusion is a contradiction. In fact, however, this paradox involves a more difficult problem that cannot be solved by either paraconsistent logic or contraction-free logic. The problem is that we can derive a negation of an arbitrary sentence. Because this problem is not triviality in a strict sense, but is similar to it, I call it "moderate triviality".

Moderate triviality is derived as follows: suppose that x is ineffable. Now *x* should not be expressed by any expression; that is, any expression should not hold if it is attributed to *x*. "Any expression" includes "*x* is P or Kyoto is in Japan". It follows that neither "*x* is P" nor "Kyoto is in Japan" holds; hence, it is not the case

[22] From it, we can say that broad self-reference is related to quantifier; but please note that not every paradox related to a quantifier is a broad self-referential paradox. Cf. Yablo's paradox (a narrow self-referential paradox).

that Kyoto is in Japan. In the same way, for any sentence ψ, we can derive that it is not the case that ψ.

Moderate triviality can be derived another way. Suppose that x is ineffable. Then *x* should not be expressed by any expression, including "if x is P, then ψ" (for example, "if *x* is omnipotence, evil does not exist"). Now, let us choose "is equal to itself" for P; that is, "if *x* is equal to itself, ψ" does not hold. But it holds that anything is equal to itself, including *x*. That means ψ does not hold.

Let us see these arguments formally; for this, first of all, let us formalize the notion of ineffability. That x is ineffable can be interpreted as that no expression can be attributed to x. One of the simplest formalizations of this interpretation that *x* is ineffable (¬Ef(x)) is

$$\neg \mathrm{Ef}(x) := \forall P(\neg P(x)).$$ [23]

As mentioned, one conclusion of the Ineffability paradox is contradiction, as follows:

$$\dfrac{\dfrac{\dfrac{\neg\mathbf{Ef}(x)}{\forall P(\neg P(x))}\,\text{def.}}{\neg\neg\mathbf{Ef}(x)}\,\forall E \qquad \neg\mathbf{Ef}(x)}{\textbf{Contradiction!}}$$

At the same time, however, we can check that another conclusion of it is moderate triviality.

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\mathbf{Ef}(x)}{\forall P(\neg P(x))}\,\text{def.}}{\neg(\varphi(x)\vee\psi)}\,\forall E}{\neg\varphi(x)\wedge\neg\psi}\,\text{DeMorgan}}{\neg\psi}\,\wedge E \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\neg\mathbf{Ef}(x)}{\forall P(\neg P(x))}\,\text{def.}}{\neg(x=x\to\psi)}\,\forall E \qquad \dfrac{[\psi]_v}{x=x\to\psi}\,\text{Weakening}}{\bot}\,\to E}{\neg\psi}\,\to I_v$$

Importantly, (1) the conclusion is not a contradiction, but moderate triviality, ¬ψ; (2) in these arguments, we do not use contraction. These two facts imply that neither paraconsistent logic, nor contraction-free logic is helpful.

3.2 Berry's Paradox

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[23] This is my formalization, but you may think there are other interpretations and formalizations of ineffability. This paper, however, does not intend to cover all interpretations and formalizations of ineffability and check whether all such formalizations fall into moderate triviality but check whether a certain formalization can be a counterexample of existing studies.

Berry's paradox (or the Berry paradox) is caused by expressions like "the smallest positive real number not expressed by under 58 letters". There are only finite expressions that consist of fewer than 58 letters because there are only 26 letters in the English alphabet and 10 letters in Arabic figures. On the other hand, there are infinitely many positive real numbers. Hence, there should be a smallest positive real number not expressed by under 58 letters. Let us call this number x. The expression "the smallest positive real number not expressed by under 58 letters", however, consists of 57 letters, so it is derived that x is and is not the smallest positive real number not expressed by under 58 letters.

3.2.1. Modified Berry's Paradox

Let us modify this paradox. Consider the expression "not expressed by under 10,000 letters". Because there are only finite expressions that consist of fewer than 10,000 letters (for the same reason as above), and there are infinitely many things in the world (at least the total number of real numbers is infinite), there should be things that satisfy the expression above. Suppose x satisfies this expression. Because this expression consists of 34 letters, it is derived that x is expressed by under 10,000 letters. Hence, again, we fall into a contradictory conclusion that x is and is not expressed by under 10,000 letters. Let us call this version the "modified Berry's paradox".

3.2.2. Is It a Self-referential Paradox?

You may also hesitate to treat this paradox as self-referential according to the same two points as in the case of the Ineffability paradox, but such doubt can be ignored for the same reasons.

Firstly, this paradox can disappear by insisting that there is no object not expressed by under 10,000 letters. We observed that it is difficult to deny something not expressed. How much more difficult to deny something not expressed by under 10,000 words!

Next, let us check its self-reference. Again, it is not a narrow self-reference, but a broad one. "X is not expressed by under 10,000 words" implies that it does not hold "x is not expressed by under 10000 words", "x is a prime number" and so on.

"Is not expressed by under 10,000 words" refers to negation of any expression less than 10,000 words about x (including "is not expressed by under 10,000 words").

3.2.3. How It Slips from Existing Works

It seems again that the modified Berry's paradox can be solved by paraconsistent logic because its conclusion is a contradiction. This paradox is also, however, concerned with another problem besides contradiction. The problem is in fact not moderate triviality, which occurs in the Ineffability paradox, but is almost the same as moderate triviality. This problem occurs as follows: x, which satisfies "not expressed by under 10,000 letters" should not be expressed by an expression like "x is a prime number" or "Kyoto is in Japan" because this expression consists of 31 letters. It follows that neither "x is a prime number" nor "Kyoto is in Japan" holds; that is, it is not the case that Kyoto is in Japan. Or, when x satisfies the property above, x should not be expressed as "if x is equal to itself, Kyoto is in Japan", which consists of 32 letters. It is not the case that if *x* is equal to itself, Kyoto is in Japan, but everything, including *x*, is equal to itself, so we derive the conclusion that it is not the case that Kyoto is in Japan. In this way, for any sentence ψ that consists of not so many letters, we can derive negation of sentence ψ; we can derive that it is not the case that ψ. True, if ψ consists of over 10,000 letters, a disjunction/conditional of ψ and some sentence that contains *x* should consist of over 10,000 letters. In such case, it does not hold that *x* should not be expressed by the disjunctive/conditional sentence; hence ¬ψ is not derived. Also true, however, is that this is not very important. First of all, almost all sentences in ordinary life consist of fewer than 10,000 letters. Moreover, you can modify the paradox to an extreme case like, "not expressed by under 1,000,000,000,000,000,000,000,000,000 letters". Anyway, although it is true that the modified Berry's paradox does not involve moderate triviality itself, no doubt the paradox does involve a problem that closely resembles moderate triviality (this problem can be called "weak moderate triviality" if you want).

Let us prove weak moderate triviality formally. That *x* is not expressed by under 10,000 letters means that for every expression φ, if φ consists of under 10,000 letters, it does not hold that φ. So "x is not expressed by under 10,000 letters ($\neg Ex_{10000}(x)$)" can be formalized as

$$\neg Ex_{10000}(x) := \forall P(P \in \Gamma \rightarrow \neg P(x))$$

(Γ means "consists of under 10,000 letters"). The modified Berry's paradox can cause a contradictory conclusion on the one hand.

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\mathbf{Ex}_{10000}(x)}{\forall P(P\in\Gamma\to\neg P(x))}\,\text{def.}}{\neg\mathbf{Ex}_{10000}(x)\in\Gamma\to\neg\neg\mathbf{Ex}_{10000}(x)}\,\forall E\qquad \neg\mathbf{Ex}_{10000}(x)\in\Gamma}{\neg\neg\mathbf{Ex}_{10000}(x)}\,\to E\qquad \neg\mathbf{Ex}_{10000}(x)}{\textbf{Contradiction!}}\,\to E$$

On the other hand, however, weak moderate triviality is also derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\mathbf{Ex}_{10000}(x)}{\forall P(P\in\Gamma\to\neg P(x))}\,\text{def.}}{(\varphi(x)\vee\psi)\in\Gamma\to\neg(\varphi(x)\vee\psi)}\,\forall E\qquad (\varphi(x)\vee\psi)\in\Gamma}{\neg(\varphi(x)\vee\psi)}\,\to E}{\neg\varphi(x)\wedge\neg\psi}\,\text{De Morgan}}{\neg\psi}\,\wedge E$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\mathbf{Ex}_{10000}(x)}{\forall P(P\in\Gamma\to\neg P(x))}\,\text{def.}}{(x=x\to\psi)\in\Gamma\to\neg(x=x\to\psi)}\,\forall E\qquad (x=x\to\psi)\in\Gamma}{\neg(x=x\to\psi)}\,\to E\qquad \dfrac{[\psi]_v}{x=x\to\psi}\,\text{Weakening}}{\bot}\,\to E}{\neg\psi}\,\to I_v$$

(Again, please note that this ψ is not arbitrary, but almost arbitrary). Because the conclusion is not contradiction, but moderate triviality, paraconsistent logic does not work well. And because we do not use contraction, contraction-free logic is also not helpful.

4. Conclusion

In this paper, we saw the brief summary of studies about self-referential paradoxes from the comprehensive approach by constructing their general structure, and observed that the Ineffability paradox and the modified Berry's paradox slip from these results. It does not follow, however, that the comprehensive approach is hopeless. It just means that such studies are concerned only with narrow self-referential paradoxes, but these two paradoxes are broad self-referential paradoxes.

It may be not so bad to concentrate only on narrow self-referential paradoxes and ignore broad self-referential paradoxes. As I said, however, our intuition says broad self-referential paradoxes are self-referential paradox-like. Hence, I recommend you consider such paradoxes as self-referential.

Then, how should a comprehensive study of (both) self-referential paradoxes be conducted? My suggestion is (1) to find general conditions of broad self-referential paradoxes, and (2) to combine them with general conditions of narrow self-referential paradoxes. Then we can obtain genuine general conditions of self-referential paradoxes and find genuine general solution(s) to them.

